Discrete Mathematics 17 (1977) 155–159.

A CHARACTERIZATION OF HYPERCUBES

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Received 25 March 1976
Revised 14 October 1976

A combinatorial characterization of hypercubes is given. A hypercube is defined as the undirected Hasse diagram of a Boolean lattice. The method used is inductive extension of partial isomorphisms.

All structures considered in this paper are assumed to be finite. The graphs do not have loops or multiple edges.

Let us recall that in a partially ordered set $(P, \leq)$ an element y is said to cover an element x, if $x < y$, but there is no z with $x < z < y$. The Hasse diagram of $(P, \leq)$ is the directed graph D whose vertices are the elements of P, and in which there is an arrow from x to y if and only if x is covered by y. Then for any $x, y \in P$, we have $x \leq y$ if and only if there is in D a directed path from x to y. A partial order is therefore determined by its Hasse diagram.

Proposition 1. *A directed graph D is the Hasse diagram of a partial order if and only if the following holds:*

Whenever there is an arrow α from a vertex x to a vertex y, α is the only directed path from x to y and there exists no directed path from y to x.

The proof is omitted.

Let D be the Hasse diagram of a partial order $(P, \leq)$. The undirected Hasse diagram of $(P, \leq)$ is the graph obtained from D by considering the arrows as edges without orientation. The characterization of undirected Hasse diagrams seems to be an unsolved problem.

Let S be a set having n elements, $|S| = n$. Let the set $\mathcal{P}(S)$ of its 2^n subsets be ordered by inclusion. The partially ordered set $(\mathcal{P}(S), \subseteq)$ is called a Boolean lattice. Obviously, its structure is determined by the number n.

Definition 1. A graph isomorphic to the undirected Hasse diagram of a Boolean lattice $(\mathcal{P}(S), \subseteq)$ is called a hypercube.

Let us consider the undirected Hasse diagram of a Boolean lattice $(\mathcal{P}(S), \subseteq)$. Let $(A_0, A_1, \ldots, A_n)$ be any path between two vertices X and Y, $A_0 = X$, $A_n = Y$. The

A_i's are subsets of S. Every A_{i+1} is obtained from A_i either by dropping an element of A_i or by adding to it a new element. Call p_{i+1} the element involved, should it be dropped from A_i or added to it. Thus the path $(A_0, A_1, \ldots, A_n)$ defines a sequence of points $(p_1, \ldots, p_n)$. Two different paths between X and Y define different sequences. If we follow a path from X to Y, each point of $X \setminus Y$ must be dropped at least once and each point of $Y \setminus X$ added. Therefore, the length of any path between X and Y is at least the cardinality of the Boolean sum $X + Y = (X \setminus Y) \cup (Y \setminus X)$, which is also equal to $|X \cup Y| - |X \cap Y|$. On the other hand, let $(p_1, \ldots, p_n)$ be any permutation of the points of $X + Y = (X \cup Y) \setminus (X \cap Y)$. Define a sequence $A_0, A_1, \ldots, A_n$ of subsets of S as follows:

$$A_0 = X,$$

$$A_{i+1} = A_i \setminus \{p_{i+1}\} \quad \text{if } p_{i+1} \in X,$$

$$A_{i+1} = A_i \cup \{p_{i+1}\} \quad \text{if } p_{i+1} \in Y.$$

$(A_0, A_1, \ldots, A_n)$ is then a path from X to Y, and it is in fact a geodesic path because its length is $|X + Y|$. It follows that the distance between X and Y is $|X + Y|$. Moreover, there is no difficulty in verifying that the correspondence between geodesics from X to Y, and permutations of $X + Y$ is bijective. We have proved the following

Proposition 2. *A hypercube is always connected. If $d(x, y)$ denotes its distance function, then for each pair of vertices x and y the number of distinct geodesics between x and y is $d(x, y)!$.*

It will be convenient to have slightly different formulations of the above property.

Proposition 3. *Let G be a connected graph with distance function $d(x, y)$. The following three conditions are equivalent:*

(1) *For each pair of vertices x and y the number of geodesics between x and y is $d(x, y)!$.*

(2) *For each pair of vertices x and y, and for any $0 \leq i \leq d(x, y)$, the number of vertices z for which $d(x, z) = i$ and $d(z, y) = d(x, y) - i$ is precisely $\binom{d(x,y)}{i}$.*

(3) *For each pair of vertices x and y, the number of those neighbours of x that lie on at least one geodesic between x and y, is precisely $d(x, y)$.*

Proof. (1) $\Longrightarrow$ (2). Assume (1). Let x and y be vertices and let $0 \leq i \leq d(x, y)$. On each geodesic between x and y there lies exactly 1 vertex z with $d(x, z) = i$ and $d(z, y) = d(x, y) - i$. On the other hand, given such a vertex z the union of a geodesic P between x and z and a geodesic Q between z and y is always a geodesic between x and y. Therefore, the number of geodesics between x and y containing a given z is $i!(d(x, y) - i)!$. Consequently, if n is the number of vertices z with $d(x, z) = i$ and $d(z, y) = d(x, y) - i$, then $n \cdot i!(d(x, y) - i)! = d(x, y)!$, *id est*

$$n = \frac{d(x,y)!}{i!(d(x,y)-i)!} = \binom{d(x,y)}{i}.$$

(3) follows from (2) by setting $i = 1$.

Finally, if we assume (3), (1) can be proved by induction on $d(x, y)$.

Proposition 4. *A hypercube is always bipartite.*

Proof. Let $(\mathcal{P}(S), \subseteq)$ be a Boolean lattice. Define

$$V_1 = \{x \in \mathcal{P}(S) \mid |x| \text{ is even}\},$$

$$V_2 = \{x \in \mathcal{P}(S) \mid |x| \text{ is odd}\}.$$

Obviously $V_1 \cup V_2 = \mathcal{P}(S)$, $V_1 \cap V_2 = \phi$ and for $i = 1$ or 2 no two vertices of V_i are adjacent in the Hasse diagram of $(\mathcal{P}(S), \subseteq)$.

Let now G be a connected bipartite graph with distance function $d(x, y)$. If, for three vertices, o, x and y of G, we have $d(o, x) = d(o, y)$ then x and y cannot be adjacent. Let us fix arbitrarily a vertex o. We define an orientation of G. Let (x, y) be an edge. We have either $d(o, x) + 1 = d(o, y)$ or $d(o, y) + 1 = d(o, x)$. Orient the edge from x towards y in the first case, from y towards x in the second. It is easy to see, using Proposition 1, that the following is true.

Proposition 5. *The graph G with the above defined orientation is the Hasse diagram of a partial order G^* whose minimum is o.*

With G as above, for each nonnegative integer k let R_k be the set of vertices of G whose distance from o is k. R_0 is reduced to o, R_1 is the neighbourhood of o, etc. Let $G_k = R_0 \cup \ldots \cup R_k$. Each G_k is a lower ideal of the partial order G^* and obviously $G_k \subseteq G_{k+1}$.

On the other hand, let $\mathcal{P}_k = \{A \in \mathcal{P}(R_1) \mid |A| \leq k\}$ for each nonnegative integer k. Each $\mathcal{P}_k$ is a lower ideal of the Boolean lattice $(\mathcal{P}(R_1), \subseteq)$, and again $\mathcal{P}_k \subseteq \mathcal{P}_{k+1}$.

Proposition 6. *If G verifies the conditions of Proposition 3, then for each nonnegative integer k the lower ideal G_k of G^* is order-isomorphic to the lower ideal $\mathcal{P}_k$ of $(\mathcal{P}(R_1), \subseteq)$.*

Proof. We use induction on k. Assume that the conditions of Proposition 3 hold.

(1) The isomorphism $G_0 \simeq \mathcal{P}_0$ is obvious.

(2) Since $\mathcal{P}_1$ contains only the empty set and the singletons of R_1, we also have $G_1 \simeq \mathcal{P}_1$.

(3) Let x and y be distinct vertices in R_1. $d(x, y) = 2$, therefore there exists a unique vertex z in R_2 adjacent to both x and y. By condition (1) of Proposition 3, if $\{x, y\} \neq \{x', y'\}$ then the corresponding vertices z and z' in R_2 are distinct because $2! = 2$. For the same reason, every vertex $z \in R_2$ corresponds to a 2-set $\{x, y\}$. Consequently, $G_2 \simeq \mathcal{P}_2$.

(4) Let $k \geq 3$. Suppose that the proposition is true for lesser values of k. Let $\psi : \mathscr{P}_{k-1} \cong G_{k-1}$ be an order-isomorphism.

Take any k-subset A of R_1. Let $A_1, \ldots, A_k$ be the subsets of A having $k-1$ elements. Let $a_i = \psi(A_i)$ for each $1 \leq i \leq k$. If $a_i \neq a_j$, then $d(a_i, a_j) = 2$, and there exists a unique vertex $s(a_i, a_j)$ in R_k adjacent to both a_i and a_j. We claim that all the $s(a_i, a_j)$ are the same. It is enough to show that for each $i \neq 1, 2$ a_i is adjacent to $s(a_1, a_2)$. If it is not true for an a_i, then $s(a_1, a_2)$, $s(a_1, a_i)$, $s(a_2, a_i)$ are pairwise distinct. G_{k-1} is isomorphic to a lower order-ideal of a lattice, therefore the "greatest lower bound" operation is well-defined in G_{k-1}: we denote it by $\wedge$. Clearly, $a_1 \wedge a_2$, $a_1 \wedge a_i$ and $a_2 \wedge a_i$ are pairwise distinct. Moreover, consider $b = a_1 \wedge a_2 \wedge a_i$. Since $d(a_1 \wedge a_i, s(a_1, a_2)) = 2$, there is a vertex $y_1 \neq a_1$ in R_{k-1}, adjacent to $a_1 \wedge a_i$ and to $s(a_1, a_2)$. Similarly, there is a vertex $y_2 \neq a_2$ in R_{k-1}, adjacent to both $a_2 \wedge a_i$ and $s(a_1, a_2)$. Using the isomorphism ψ, it is easily checked that y_1, a_1, a_i, a_2, y_2 are pairwise distinct (see Fig. 1).

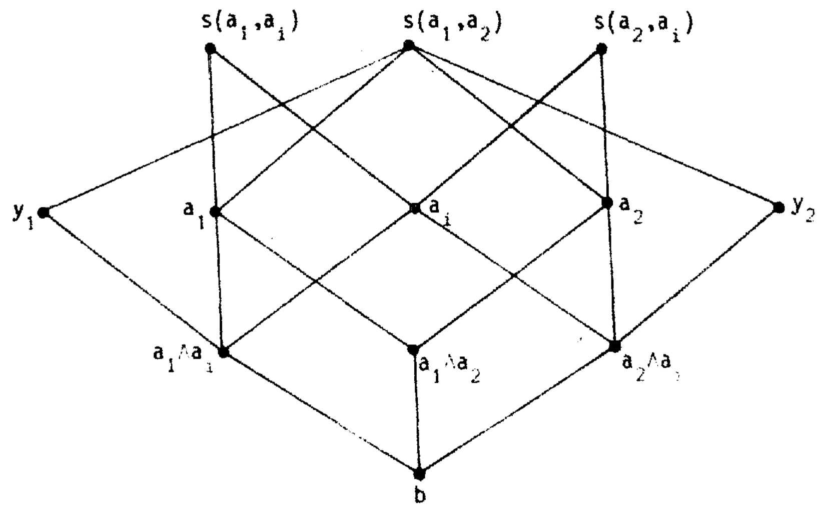

Fig. 1.

The four neighbours a_1, a_2, y_1, y_2 of $s(a_1, a_2)$ lie each on a geodesic between $s(a_1, a_2)$ and b. Since $d(s(a_1, a_2), b) = 3$, this contradicts condition (3) of Proposition 3. Therefore, all the $s(a_i, a_j)$ must be identical, as claimed. We define an extension $\bar{\psi}$ of ψ to $\mathscr{P}_k$ by letting $\bar{\psi}(A)$ be this unique vertex $s(a_i, a_j)$, and proceeding similarly for every k-subset of R_1.

We have to verify that $\bar{\psi}$ is an isomorphism between $\mathscr{P}_k$ and G_k. For each k-subset A of R_1, $\bar{\psi}(A)$ is in R_k. In order to show that $\bar{\psi}$ is injective, it is enough to check that for distinct k-subsets A and A' of R_1 we also have $\bar{\psi}(A) \neq \bar{\psi}(A')$. But this follows from condition (3), applied with $x = \bar{\psi}(A)$ and $y = 0$. Similarly, to prove that $\bar{\psi}$ is surjective onto G_k, we have only to show that each vertex $x \in R_k$ is in the image of $\bar{\psi}$. Let indeed a be a neighbour of x in R_{k-1} and y a neighbour of a in R_{k-2}. Since $d(x, y) = 2$, there must exist an other vertex $a' \neq a$ in R_{k-1} adjacent to both x and y. x must be $\bar{\psi}[\psi^{-1}(a) \cup \psi^{-1}(a')]$.

It is equally easy to verify that $\bar{\psi}$ preserves the covering relation and so does $\bar{\psi}^{-1}$. This completes the proof that $\bar{\psi}$ is an isomorphism between $\mathcal{P}_k$ and G_k.

Let us observe that for k sufficiently large, $\mathcal{P}_k = \mathcal{P}(R_1)$ and G_k exhausts the set of vertices of G. Applying Proposition 6 to such a k, we get:

Proposition 7. *$G^{\leq}$ is order-isomorphic to $(\mathcal{P}(R_1), \subseteq)$.*

Proposition 7 proves that G is the undirected Hasse diagram of a Boolean lattice, *id est* a hypercube. In our construction we used the assumption that the connected graph G satisfied the conditions of Proposition 3 and that G was bipartite. (It would be desirable to know whether the assumption "G is bipartite" can be replaced by the weaker assumption "G does not contain a triangle".) In view of Propositions 2 and 4, we have the following characterization of hypercubes:

Proposition 8. *A connected graph G with distance function $d(x, y)$ is a hypercube if and only if the following two conditions are satisfied:*

(i) *G is bipartite,*

(ii) *for each two vertices x and y of G the number of geodesics between x and y is $d(x, y)!$.*

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